PMM U.S.S.R.,Vol.45,pp.317-320
0021-8928/82/3 0317 $7.50/0

UDC 531.36

# ON THE PROBLEM OF STABILITY IN THE CRITICAL CASE*

V.N. TKHAI

A critical case of $m$ zero roots, $n$ pairs of pure imaginary roots and $q$ roots with negative real parts in which $m$ groups of solutions correspond to the $m$-tuple zero root and no integral relations exist between the pure imaginary roots is considered, and a case in which asymptotic stability is impossible is singled out. The Kamenkov theorem on stability is extended to an essentially singular case. A theorem on stability for the nonessentially singular cases is established for the critical case of a single zero root, arbitrary number of pairs of pure imaginary roots and roots with negative real parts.

Let us consider a critical case of $m$ zero roots, $n$ pairs of pure imaginary roots and $q$ roots with negative real parts. Let $m$ groups of solutions correspond to the $m$-tuple zero root, and let no integral relations exist between the pure imaginary roots. We shall assume that the equations of perturbed motion have holomorphic right-hand sides. Then, separating the explicitly linear approximation we have

$$\begin{aligned} &\xi^{\cdot} = F_0 + F_1,\ \ \eta^{\cdot} = \Lambda\eta + \Phi_0 + \Phi_1,\ \ \overline{\eta}^{\cdot} = -\Lambda\overline{\eta} + \overline{\Phi}_0 + \overline{\Phi}_1, \\ &z^{\cdot} = Pz + Z_0 + Z_1,\ \ \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \end{aligned} \tag{1}$$

Here $\xi$ and $z$ are real $m$- and $q$-vectors, $\eta$ and $\overline{\eta}$ are the complex conjugate $n$-vectors, $\overline{\Phi}_0, \overline{\Phi}_1$ are complex vector functions conjugated with $\Phi_0$ and $\Phi_1$ respectively, $\Lambda$ is the diagonal matrix of pure imaginary quantities, and the constant $(q \times q)$ matrix $P = \| p_{sj} \|$ has eigenvalues with negative real parts. The functions with subscript 1 and identically equal to zero when $z = 0$. The arguments of the functions with subscript 0 and 1 are $\xi, \eta, \overline{\eta}$, and $\xi, \eta, \overline{\eta}, z$ respectively.

According to the Liapunov theorem on existence of holomorphic functions satisfying partial differential equations /1/, we can always arrive, by means of a nondegenerate transformation, at a situation in which the functions with a subscript 0 do (do not) vanish identically simultaneously everywhere. Moreover, if an essentially singular case

$$F_0 = \Phi_0 = \overline{\Phi}_0 = Z_0 \equiv 0 \tag{2}$$

exists, then the zero solution of the system (1) is Liapunov stable /2/. We see that, when (2) holds, the equations (1) admit $(2n + m)$-parameter family of particular solutions ($c$ is a constant $m$-vector)

$$\xi = c,\ \ \eta^{\cdot} = \Lambda\eta,\ \ \overline{\eta}^{\cdot} = -\Lambda\overline{\eta},\ \ z = 0 \tag{3}$$

The assertion concerning the stability can be proved with help of the function /2/

$$V = \sum_{i=1}^{m} \xi_i^2 + \sum_{j=1}^{n} \eta_j \overline{\eta}_j + W(z_1, \ldots, z_q)$$

where the quadratic form $W$ is determined from the equation

$$\sum_{s=1}^{q} \frac{\partial W}{\partial z_s} \sum_{j=1}^{q} p_{sj} z_j = -(z_1^2 + \ldots + z_q^2) \tag{4}$$

Indeed, when (2) holds, using a suitable transformation we can always transform the functions $F_1, \Phi_1, \overline{\Phi}_1$ to the form not containing terms linear relative to $z_1, \ldots, z_q$ /2/. This means that the total derivative of the function $V$ is, by virtue of (1),

$$V^{\cdot} < -\mu (z_1^2 + \ldots + z_q^2),\ \ \mu - \text{const},\ 0 < \mu < 1$$

*Prikl.Matem.Mekhan.,45,No.3,436-440,1981 and all conditions of the Liapunov theorem on stability /1/ hold.

From the stability with respect to all variables $\xi, \eta, \bar{\eta}, z$ it follows that the function $W$ proves, in accordance with the Rumiantsev theorem /3/, asymptotic $z$-stability. Moreover, since the function $W$ is chosen as a quadratic form, we have

$$|z_s| < \varepsilon e^{-\alpha t} \ (s = 1, \ldots, q)$$

where $\varepsilon$ and $\alpha$ are positive constants. Consequently the following estimates hold:

$$|F_1| < \varepsilon_1 e^{-2\alpha t}, \ |\Phi_1| < \varepsilon_1 e^{-2\alpha t}, \ |\bar{\Phi}_1| < \varepsilon_1 e^{-2\alpha t}$$

where $\varepsilon_1$ is a positive constant, i.e.

$$|\xi^\cdot| < \varepsilon_1 e^{-2\alpha t}, \ |\eta^\cdot - \Lambda\eta| < \varepsilon_1 e^{-2\alpha t}, \ |\bar{\eta}^\cdot + \Lambda\bar{\eta}| < \varepsilon_1 e^{-2\alpha t}$$

and every perturbed motion tends asymptotically to one of the solutions given by (3).

Thus we have shown that in the essentially singular case considered here the variables change in the same manner as in the Liapunov—Malkin theorem /4/ with the only difference, that here every perturbed motion tends asymptotically to some periodic solution of the $(2n + m)$-parameter family (3).

Let us assume now that the indentities (2) do not hold. We shall write the functions with zero subscript in the form of a sum of two terms:

$$F_0 = F_0^{(1)}(\xi) + F_0^{(2)}(\xi, \eta, \bar{\eta}), \quad \Phi_0 = \Phi_0^{(1)}(\xi) + \Phi_0^{(2)}(\xi, \eta, \bar{\eta})$$
$$\bar{\Phi}_0 = \bar{\Phi}_0^{(1)}(\xi) + \bar{\Phi}_0^{(2)}(\xi, \eta, \bar{\eta},), \quad Z_0 = Z_0^{(1)}(\xi) + Z_0^{(2)}(\xi, \eta, \bar{\eta})$$
$$F_0^{(2)}(\xi, 0, 0) = \Phi_0^{(2)}(\xi, 0, 0) = \bar{\Phi}_0^{(2)}(\xi, 0, 0) = Z_0^{(2)}(\xi, 0, 0) \equiv 0.$$

According to the Liapunov theorem on existence of holomorphic functions satisfying partial differential equations, and its supplement due to Kamenkov /2/, using a nondegenerate transformation we can always transform the functions with superscript 1 to the form vanishing (not vanishing) identically, simultaneously everywhere. If the conditions

$$F_0^{(1)} = \Phi_0^{(1)} = \bar{\Phi}_0^{(1)} = Z_0^{(1)} \equiv 0 \tag{5}$$

hold, then the equations (1) admit a family of steady state motions

$$\xi = c, \ \eta = 0, \ \bar{\eta} = 0, \ z = 0$$

and the asymptotic stability is impossible. (We note that such a situation arises in nonholonomic systems /5/. Moreover, we can alsways transform the functions $F_0^{(2)}$ and $F_1$ to the form not containing terms linear in $\eta, \bar{\eta}$ and $z$ /2/.

Let us consider the case $m = 1$ in more detail. The sufficient conditions of the asymptotic stability and instability are obtained in /6/. The arguments used there embrace the cases in which the conditions (5) do not hold. Let us assume that they do hold. Then, using the polar coordinates

$$\eta_s = \rho_s \exp(i\theta_s), \ \bar{\eta}_s = p_s \exp(-i\theta_s) \ (s = 1, \ldots, n)$$

we can write the system (1) in the form /6/

$$x^\cdot = \sum_{j=1}^{n} a_j \rho_j^2 + X(x, \rho, \theta, z) \tag{6}$$
$$\rho_s^\cdot = b_s x \rho_s + R_s(x, \rho, \theta, z) \ (s = 1, \ldots, n)$$
$$z_l^\cdot = \sum_{j=1}^{q} p_{lj} z_j + Z_l(x, \rho, \theta, z) \quad (l = 1, \ldots, q)$$

where $x$ is a real variable, $a_j$ and $b_s$ are constant coefficients accompanying the quadratic terms and the expansions of the functions $X$ and $R_s$ into series in the variables $x, \rho$ and $z$ begin with the terms of at least third order. In addition, by virtue of (5) we have

$$X(x, 0, \theta, 0) = R_s(x, 0, \theta, 0) = Z_l(x, 0, \theta, 0) \equiv 0$$

and the function $X$ contains no terms linear in $\rho$ and $z$.

If amongst $a_s$, $b_s$ $(s = 1, \ldots, n)$ a pair $a_{s_*}$, $b_{s_*}$ of the same sign exists, then the zero solution $x = \rho = z = 0$ in Liapunov unstable /6/. Let us assume that $a_s b_s < 0$ $(s = 1, \ldots, n)$. In the $(n + 1)$-dimensional spherical coordinates

$$x = r\cos\varphi_1, \rho_s = r\cos\varphi_{s+1}\prod_{j=1}^{s}\sin\varphi_j \quad (s = 2, \ldots, n-1) \tag{7}$$

$$\varphi_n = r\prod_{j=1}^{n}\sin\varphi_j, \ 0 \leqslant \varphi_1 \leqslant \pi, \ 0 \leqslant \varphi_j \leqslant \pi/2 \quad (j = 2, \ldots, n)$$

the problem of stability of the solution $x = \rho = z = 0$ of the system (6) is reduced to the problem of stability of the solution $r = z = 0$ of the system

$$r^{\cdot} = \cos\varphi_1 X^* + \sum_{i=1}^{n-1}\Big(\cos\varphi_{i+1}\prod_{j=1}^{i}\sin\varphi_j\Big)R_i^* + \Big(\prod_{j=1}^{n}\sin\varphi_j\Big)R_n^* \tag{8}$$

$$r\varphi_1^{\cdot} = r^2 Q_1\sin\varphi_1 + \cos\varphi_1\Big[\sum_{i=1}^{n-1}\Big(\cos\varphi_{i+1}\prod_{j=1}^{i}\sin\varphi_j\Big)R_i^* + \Big(\prod_{j=1}^{n}\sin\varphi_j\Big)R_n^*\Big] - \sin\varphi_1 X^*$$

$$\Big(\prod_{j=1}^{s-1}\sin\varphi_j\Big)\varphi_s^{\cdot} = \ldots \quad (s = 2, \ldots, n)$$

$$z_l^{\cdot} = \sum_{j=1}^{q} p_{lj}z_j + Z_l^*(r, \varphi, \theta, z) \quad (l = 1, \ldots, q)$$

$$Q_1 = b_1\cos^2\varphi_2 + b_2\cos^2\varphi_3\sin^2\varphi_2 + \ldots + b_{n-1}\cos^2\varphi_n\prod_{j=2}^{n-1}\sin^2\varphi_j + b_n\prod_{j=2}^{n}\sin^2\varphi_j$$

(we assumed here, without loss of generality, that $a_s = -b_s$ $(s = 1, \ldots, n)$, $X^*$, $R_s^*$, and $Z_l^*$ are functions of $X$, $R_s$ and $Z_l$ after the substitution (7)) with respect to the variables $r$ and $z$. The derivative of the function

$$V = r\exp(h\cos\varphi_1) + W, \quad h = \text{const}$$

($W$ is found from (4)) has, by virtue of the equations (8), the form

$$\begin{aligned} V^{\cdot} = {} & \exp(h\cos\varphi_1)\Big[\cos\varphi_1 X^* + \sum_{i=1}^{n-1}\Big(\cos\varphi_{i+1}\prod_{j=1}^{i}\sin\varphi_j\Big)R_i^* + \\ & \Big(\prod_{j=1}^{n}\sin\varphi_j\Big)R_n^*\Big] - hr^2\sin^2\varphi_1 Q_1 - h\exp(h\cos\varphi_1)\sin\varphi_1 \times \\ & \Big\{\cos\varphi_1\Big[\sum_{i=1}^{n-1}\Big(\cos\varphi_{i+1}\prod_{j=1}^{i}\sin\varphi_j\Big)R_i^* + \Big(\prod_{j=1}^{n}\sin\varphi_j\Big)R_n^*\Big] - \\ & \sin\varphi_1 X^*\Big\} - (z_1^2 + \ldots + z_q^2) + \sum_{l=1}^{q}\frac{\partial W}{\partial z_l}Z_l(r, \varphi, \theta, z) \end{aligned}$$

Since the function $X$ does not contain terms linear in $\rho$ and $z$ and the functions $X$, $R_s$ and $Z_l$ vanish when $\rho = z = 0$, we can write the derivative $V^{\cdot}$ as follows:

$$V^{\cdot} = -hr^2\sin^2\varphi_1 Q_1 - (z_1^2 + \ldots + z_q^2) + r_1^2\sin^2\varphi_1\Psi(r, \varphi, \theta, z) + r\sin\varphi_1\sum_{j=1}^{q}z_j\Psi_j(r, \varphi, \theta, z) + \sum_{i=1}^{q}\sum_{j=1}^{q}\Psi_{ij}(r, \varphi, \theta, z)z_i z_j$$

where the functions $\Psi$, $\Psi_j$, $\Psi_{ij}$ vanish when $r = z = 0$. Let us assume that all $b_s$ $(s = 1, \ldots, n)$ have the same sign. Then, choosing $h$ with sign opposite to that of $b_s$, we obtain $hQ_1 >$ const $> 0$ and the function $V$ satisfies all conditions of the Rumiantsev theorem /7/ on stability with respect to the variables $r, z_1, \ldots, z_q$.

**Theorem.** If the conditions

$$a_s b_s < 0 \quad (s = 1, \ldots, n)$$

hold and all coefficients $b_s$ $(s = 1, \ldots, n)$ are of the same sign, then the zero solution $x = \rho =$

We note that the theorem provides the criteria of stability for the nonessentially singular cases.

REFERENCES

1. LIAPUNOV A.M., Collected Works, Vol.2, General problem of stability of motion. Moscow – Leningrad, Izd. Akad. Nauk SSSR, 1956.
2. KAMENKOV G.V., Collected Works, Vol.2. Stability and oscillations of mechanical systems. Moscow, NAUKA, 1971.
3. RUMIANTSEV V.V., On the optimal stabilization of controlled systems. PMM Vol.34, No.3, 1970.
4. MALKIN I.G., The Theory of Motion Stability, Moscow, NAUKA, 1966.
5. RUMIANTSEV V.V. and KAREPETIAN A.V., Stability of motions of nonholonomic systems. In the book: Results of Science and Technology. Ser. Obshchaia mekhanika, Vol.3, Moscow, VINITI, 1976.
6. MEDVEDEV S.V. and TKHAI V.N., On stability in one critical case. PMM Vol.43, No.6, 1979.
7. RUMIANTSEV V.V., On the stability of motion with respect to a part of the variables. Vestn. MGU, No.4, 1957. (See also, in English, L.—N.Y. Academic Press, 1971).

Translated by L.K.